JOURNAL OF APPROXIMATION THEORY **29**, 2–5 (1980)

Some Notes on Turán's Mathematical Work

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Communicated by Oved Shisha

I first met Turán on September 1, 1930, in the mathematical seminar room at the University of Budapest. We collaborated for 46 years and wrote 28 joint papers in various branches of mathematics. Our personal and scientific contact had never ceased except in 1942–1945, during the dark days of World War II, when I was in the United States and Turán in Hungary.

In my paper Problems in Number Theory and Combinatorics, in "Proceedings, 6th (1976) Manitoba Conference on Numerical Mathematics," pp. 35–38, I wrote a long history of our collaboration and described Turán as a mathematician and a person as I, perhaps his closest friend, saw him. I shall refer to this paper as I. Here, in this short note, I shall merely stat some of Turán's most important results. I cannot entirely help being a bit biased and may give, perhaps, undue weight to our joint work.

Turán's most important and most original work was undoubtedly his power sum method, which he and others used with great success in various branches of analytic number theory, theory of differential equations, quasi-analytic function theory, complex analysis, numerical analysis, etc. Turán and S. Knapowski created the subject of comparative prime number theory, which is based on that method. A comprehensive book of Turán's will soon appear on this subject. (The book was completed by his students G. Halász and J. Pintz after his untimely death.) My own share in this subject has been modest and is discussed in I.

Halász wrote two nice papers on Turán's power sum method (*Math. Lapok* **27** (1976–1979), (27–41, in Hungarian, and a paper in English which will soon appear in *Acta Arithmetica*). Pintz has another very nice paper on Turán's work in number theory, which, fortunately, is in English (to appear). Here I only state one of the many theorems of Turán concerning the power sum method:

Let $f(x) = \sum_{j=1}^{n} b_j e^{\lambda_j x}$ (Re $\lambda_j \geqslant 0$). Then, for every positive a and d,

$$|f(0)| \leqslant \left(\frac{2e(a+d)}{d}\right)^n \max_{an/d \leqslant \nu \leqslant an/d+n} \left|f\left(\frac{d\nu}{n}\right)\right| \leqslant \left(\frac{2e(a+d)}{d}\right)^n \max_{a \leqslant x \leqslant a+d} |f(x)|.$$

0021-9045/80/050002-04$02.00/0

This theorem has an astonishing number of applications.

Turán's main interests, since childhood were prime numbers, analytic number theory, and Riemann's function. Turán was an "unbeliever," in fact, a "pagan": he did not believe in the truth of Riemann's hypothesis. He hoped that his power sum method would lead to a decision; perhaps his hope will be fulfilled long after his great brain ceased to think.

His second great discovery was the so-called Turán–Kubilius inequality which is really an application of Tchebicheff's inequality (which Turán redisvovered in 1934): for more details see J. Kubilius, "Probabilistic Methods in the Theory of Numbers, "Amer. Math. Soc., Providence, R. I., 1964; and P. D. T. A. Elliott, "Probabilistic Number Theory." Vols. I, II, Springer-Verlag, New York, 1979.

In the 1930s, in my early work on the distribution functions of additive number-thoeretic functions, I used the methods of Brun and Turán a great deal. R. Rado once remarked that I perform black magic with these method and L. K. Hua once said, "Erdös's rice bowl is the methods of Brun and Turán."

Turán and I raised some interesting problems of a new type on the distribution of prime numbers. Perhaps our most interesting, challenging, and annoying problem is the following: Put $p_{n+1} - p_n = d_n$. Is it true that at least one of the two, (i) $d_n > d_{n+1} > d_{n+2}$ and (ii) $d_{n+2} > d_{n+1} > d_n$, has infinitely many solutions? For more details see I.

Turán and I worked jointly for many years on interpolation. Our first great success occurred in 1934 when we proved that, for very general point groups, the sequence of Lagrange interpolation polynomials converges in the mean. It is hard to understand why this was not discovered much earlier. In general we were very lucky in our work, most of which was new; we had bad luck only twice. We "discovered" in 1934 that, for every point group, there is a continuous function $f(x)$ and a point x_0, $-1 < x_0 < 1$, such that the sequence of Lagrange interpolation polynomials $\mathscr{L}_n(f(x))$ diverges at x_0. We soon found out that S. Bernstein anticipated us by a few years. Our second piece of bad luck was the rediscovery of some theorems of W. Markoff concerning extremal problems for polynomials.

Turán and I have several papers on the discrepancy of sequences. Here are two of our results; they have had considerable influence and have been quoted a great deal. Let $a_0 + \cdots + a_n z^n$ $(a_0 a_n \neq 0)$ be a polynomial with zeros $z_\nu = |z_\nu| e^{i\phi_\nu}$, $\nu = 1, \ldots, n$. Then

$$\left| \sum_{\alpha<\phi_\nu<\beta} 1 - \frac{(\beta - \alpha) n}{2\pi} \right| < 16 \sqrt{n \left| \log \frac{\sum_{i=0}^{n} |a_i|}{\sqrt{|a_0 a_n|}} \right|} \tag{1}$$

(On the Distribution of Roots of Polynomials, *Ann. of Math.* **51** (1950), 105–111). Our original proof has been improved by T. H. Ganelius by using methods of potential theory. The problem remains whether an estimate like (1) remains true if the degree n is replaced by the number of nonvanishing terms. We hoped to return to this subject, but Turán's untimely death prevented us.

Our second result has had a greater impact:

Put

$$s_k = \left| \sum_{\nu=1}^{n} e^{ki\phi_\nu} \right| \leqslant \psi(k), \qquad k = 1,\ldots, m.$$

Then

$$\left| \sum_{\alpha \leqslant \phi_\nu \leqslant \beta} 1 - \frac{\beta - \alpha}{2\pi} n \right| < C \left[\frac{n}{m+1} + \sum_{k=1}^{m} \frac{\psi(k)}{k} \right] \tag{2}$$

Result (2) strengthened a previous result of J. F. Koksma and J. G. van der Corput (On a Problem in the Theory of Uniform Distribution, *Indag. Math.* **10** (1948), 3–19).

In fact, Turán and I have several much sharper results, assuming $|l_k(x)| < C$, $-1 < x < 1$, where $l_k(x)$ are the fundamental functions of Lagrange interpolation. These results were almost completely ignored: the reason seems to be the fact that it is usually very hard to verify a condition $|l_k(x)| < C$.

Turán has many other results on interpolation but I must mention other subjects, too.

In the last 15 years of Turán's life we worked a great deal on statistical group theory. Both of us hoped that this subject would have a bright future and we planned further work.

Our work has already had some application: J. D. Dixon proved an old conjecture of E. Netto: if $\mathscr{S}_n$ is the symmetric group on n letters and if we choose two elements at random, then, with probability $\frac{3}{4}$, these two elements generate S_n.

Turán had the rare and remarkable ability to raise important and interesting problems in subjects far away from his own. Some of these problems have had later important applications: for more details see I. In 1940–1941 he created the area of extremal problems in graph theory which is now one of the fastest-growing subjects in combinatorics. Very recently a comprehensive book on this subject by B. Bollobás appeared ("Extremal Graph Theory," Academic Press, London, 1978). Often Turán never returned to a subject he created, but in this case, Turán, V. T. Sós (Mrs. Turán), A. Meir, and I have a series of three papers where we apply Turán's original theorem to geometry, potential theory, and other subjects.

As stated in I, Turán maintained his interest in mathematics to the end: his last words, a few hours before his death, were $O(1)$.

In the distant past kings were addressed: "0, King, may you live for ever." It is more realistic to address a mathematician: "May your theorems live forever." I hope and expect that Turán's theorems and problems will live forever.